Ocean surface waves: linear theory

Copić, Antea

University of Split
Faculty of Science

Ocean surface waves : linear theory

Bachelor thesis

Antea Copić

Split, September 2022.

Temeljna dokumentacijska kartica

Sveučilište u Splitu
Prirodoslovno – matematički fakultet
Odjel za fiziku
Ruđera Boškovića 33, 21000 Split, Hrvatska

Završni rad

Površinski valovi u moru: linearna teorija

Antea Copić

Sveučilišni preddiplomski studij Fizika

Sažetak:

Cilj ovog rada jest izvesti jednadžbe gibanja za površinske valove u moru. Krećemo od opisa općeg vala, nakon čega demonstriramo princip linearne superpozicije valova. Pokazujemo kako izgledaju matematički izrazi koji opisuju valove dubokog i plitkog mora. Valne jednadžbe izvest ćemo, polazeći od Bernoullijeve i Laplaceove jednadžbe i seta rubnih uvjeta. S obzirom na to da su u pitanju kompleksniji izrazi, koriste se razne aproksimacije s ciljem lakšeg rješavanja problema. Kao rezultat, dobiju se jednadžbe gibanja čija rješenja zadovoljavaju generalni oblik jednadžbi vala.

Ključne riječi: postulati, površinski val, jednadžbe gibanja, linearna teorija

Rad sadrži: 16 stranica, 4 slike, 0 tablica, 0 literaturnih navoda. Izvornik je na hrvatskom / engleskom jeziku

Mentor: doc. dr. sc. Žarko Kovač

Neposredni voditelj: doc. dr. sc. Žarko Kovač

Ocjenjivači: doc. dr. sc. Žarko Kovač
prof. dr. sc. Ante Bilušić
doc. dr. sc. Ivana Weber

Rad prihvaćen: 23. 09. 2022.

Rad je pohranjen u knjižnici Prirodoslovno – matematičkog fakulteta, Sveučilišta u Splitu.

Basic documentation card

University of Split
Faculty of Science
Department of Physics
Ruđera Boškovića 33, 21000 Split, Croatia

Bachelor thesis

Ocean surface waves: linear theory

Antea Copić

University undergraduate study program Physics

Abstract:

The aim of this thesis is to derive the equations of motion for surface waves in the ocean. We start from the description of a general wave, after which we demonstrate the principle of linear superposition of waves. We show what the mathematical expressions that describe deep and shallow sea waves look like. We will derive the wave equations, starting from the Bernoulli's and Laplace's equations and a set of boundary conditions. Considering that the expressions are more complex, various approximations are used with the aim of solving the problem more easily. As a result, the equations of motion are obtained, the solutions of which satisfy the general form of the wave equations

Keywords: postulates, surface wave, equations of motion, linear theory

The thesis consists of: 16 pages, 4 figures, 0 tables, 0 references. Original language: English

Supervisor: Asoc. Prof. Dr. Žarko Kovač

Leader: Asoc. Prof. Dr. Žarko Kovač

Reviewers: Asoc. Prof. Dr. Žarko Kovač
Prof. Dr. Ante Bilušić
Asoc. Prof. Dr. Ivana Weber

Thesis accepted: 23. 09. 2022.

Thesis is deposited in the library of the Faculty of Science, University of Split.

Contents

1 Introduction

In the ocean, waves are visible during any time of a day. A wave is distrubance on the surface of the water, which is caused by the wind. Sometimes, the waves are of higher amplitude, sometimes they are barely noticable.

In this thesis, we will derive equations of motion for surface waves. Each wave is best described by its amplitude, wavenumber and frequency. Amplitude of a wave is its maximum displacement from the equilibrium state. Frequency expresses the number of cycles of the repetitive waveform per second. Wavenumber is equal to the true frequency divided by the speed of the wave, and thus equal to the number of waves in a unit distance. Each wave considered here is linear, which means that it has a small amplitude in comparison to its wavelength.

Some of the ocean properties are ignored, such as viscosity, as it is not crucial for the phenomena which will be described. The ocean is considered continuous and all equations are ideal hydrodynamic equations. The starting point will be the two postulates. First postulate states that any wave can be described by the equation $\eta = Acos(kx - \omega t)$, while the second postulate allows us to add as many waves as we want, under the condition that the postulate 1 is satisfied.

In solving this linear problem, we will use many approximations and boundary conditions to simplify the problem as much as possible. In some way, we will make a full circle : starting with postulates that give us the general form of velocity components and frequency, and ending up with the solutions of the same form. All materials that were used in this thesis are listed under literature.

2 Basic wave postulates

In order to study ocean waves, one must first define a coordinate system. The main axis of focus is z-axis, as it is used to describe the depth of the ocean. Depth of $z = 0$ equals the ocean surface at steady state, while $z = -H$ equals the ocean bottom. In this system, x-axis and y-axis are both perpendicular to each other and to z-axis. Surface of the ocean is described as $z = \eta(x, y, z, t)$.

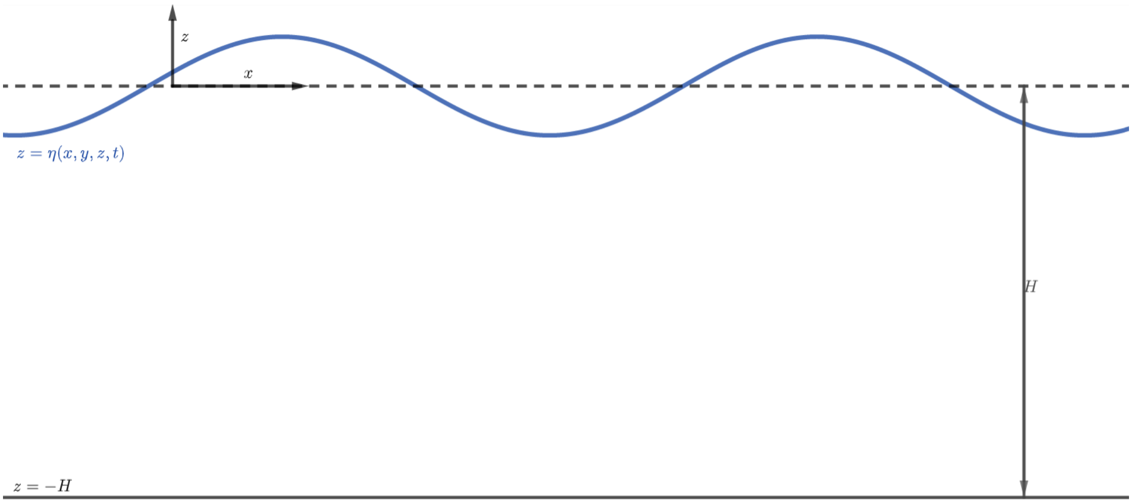

Figure 2-1:*Ocean surface and a basic wave propagating. Blue line is the surface disturbance. Direction of positive z-axis is up, while the direction of psoitive x-axis is to the right. Ocean bottom is in this case considered flat. H is the depth of the ocean.*

There are two postulates which are used to define basic waves. The first is:

Postulate 1.

If $|A|k \ll 1$, then the equation:

$$\eta = Acos(kx - \omega t), \tag{2.1}$$

describes a single, basic wave moving in the x-direction. In this equation, A is the amplitude of the wave, k is the wavenumber and ω is the frequency. All these values are constant. Relationship between ω and k is given by dispersion relation:

$$\omega = \sqrt{gktanh(kH)}. \tag{2.2}$$

Here, $g = 9.8\frac{m}{s^2}$ is the gravitational constant and the hyperbolic tangent function is defined as:

$$tanh(kH) = \frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}}. \tag{2.3}$$

The second postulate is:

Postulate 2.

If $|A|k \ll 1$, then we can add as many waves as possible, if Postulate 1 is satisfied.

One example of this motion is:

$$\eta = A_1 \cos(k_1 x - \omega_1 t) + A_2 \cos(k_2 x - \omega_2 t), \tag{2.4}$$

under the condition that (k_1, ω_1) and (k_2, ω_2) satisfy the following equations:

$$\omega_1 = \sqrt{g k_1 \tanh(k_1 H)}\ ,\ \omega_2 = \sqrt{g k_2 \tanh(k_2 H)}. \tag{2.5}$$

These two postulates will be our starting points. Other important relations between the constants are:

$$\lambda = \frac{2\pi}{|k|}, \tag{2.6}$$

where λ is the wavelength. Equation that gives the period of the wave is:

$$T = \frac{2\pi}{\omega}. \tag{2.7}$$

The phase speed c of the wave is given by:

$$c = \frac{\omega}{k}. \tag{2.8}$$

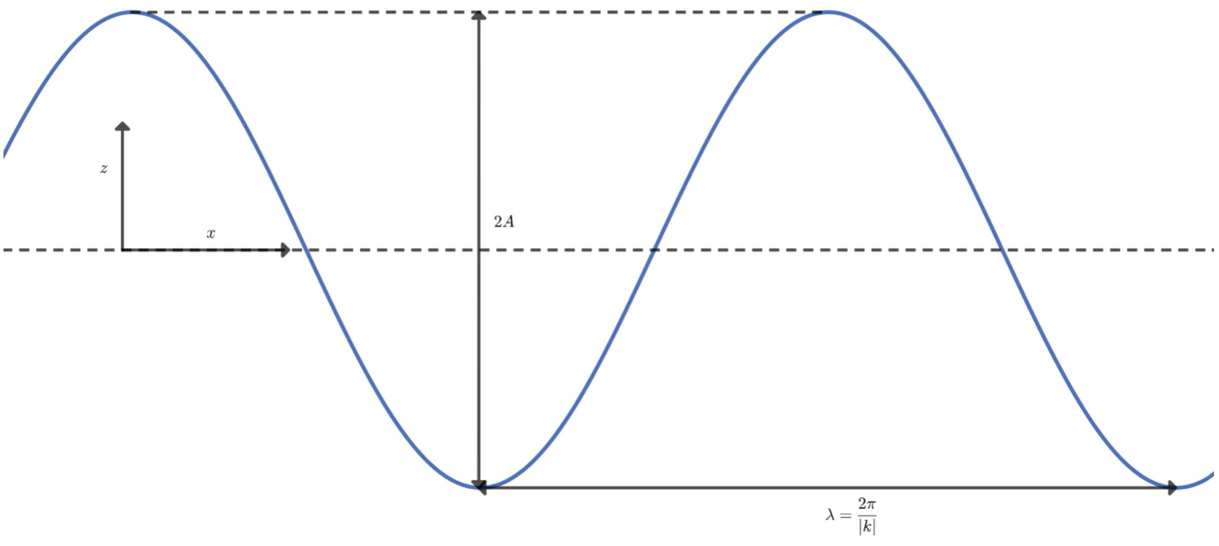

Figure 2-2:*Blue curve represents a wave propagating. In this case, positive direction of z-axis is up, and positive direction of x-axis is to the right. The wave height is twice the wave amplitude. Wavelength is the distacne between two crests troughs or any two points of the same phase on the wave.*

3 Deep water and shallow water equations

In both postulates, we mentioned the restriction $A|k| \ll 1$. This is an important restriction. It states that the wave height must be small compared to the wavelength. Small amplitude A means that we are considering linear waves. Linear theory cannot explain energy transfer between waves or wave breaking.

Equations (2.1) and (2.2) have physically incomplete description. To complete the description, we must define how the velocity of the fluid is connected with location and time. Velocity of the fluid is a vector field, depending on variables (x, y, z, t). It can be written as:

$$\vec{v}(x, y, z, t) = \big(u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)\big). \tag{3.1}$$

For the wave described by equations (2.1) and (2.2), $v = 0$. The x-componet and z-component of velocity are:

$$u = A\omega \frac{cosh\big(k(z+H)\big)}{sinh(kH)} cos(kx - \omega t), \tag{3.2}$$

$$w = A\omega \frac{sinh\big(\kappa(z+H)\big)}{sinh(kH)} sin(kx - \omega t). \tag{3.3}$$

These expressions are somewhat complex. But, if we let k be positive, we are focusing only on the waves propagating to the right. First we will focus at the case of deep water waves, by observing the limit $kH \gg 1$

$$tanh(kH) = \frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}} \to \frac{e^{kH}}{e^{kH}} = 1, \tag{3.4}$$

$$\frac{cosh\big(k(z+H)\big)}{sin(kH)} = \frac{e^{k(z+H)} + e^{-k(z+H)}}{e^{kH} - e^{-kH}} \to \frac{e^{k(z+H)}}{e^{kH}} = e^{kz}, \tag{3.5}$$

$$\frac{sin\big(k(z+H)\big)}{sin(kH)} = \frac{e^{k(z+H)} - e^{-k(z+H)}}{e^{kH} - e^{-kH}} \to \frac{e^{k(z+H)}}{e^{kH}} = e^{kz}. \tag{3.6}$$

This means that the deep water waves are described with the following equations:

$$\eta = A\, cos(kx - \omega t), \tag{3.7}$$

$$\omega = \sqrt{gk}, \tag{3.8}$$

$$u = A\omega e^{kz}\, cos(kx - \omega t), \tag{3.9}$$

$$w = A\omega e^{kz}\, cos(kx - \omega t). \tag{3.10}$$

The other case is the limit $kH \ll 1$. This is for the shallow water waves where depth H is much smaller than the wavelength λ:

$$tanh(kH) = \frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}} = \frac{(1 + kH + \cdots) - (1 - kH + \cdots)}{(1 + kH + \cdots) + (1 - kH + \cdots)} \to \frac{2kH}{2} = kH, \tag{3.11}$$

$$\frac{sin\big(k(z+H)\big)}{sin(kH)} = \frac{e^{k(z+H)} - e^{-k(z+H)}}{e^{kH} - e^{-kH}} = \frac{(1 + k(H+z) + \cdots) + (1 - k(H+z) + \cdots)}{(1 + kH + \cdots) - (1 - kH + \cdots)} \to \frac{2}{2kH} = \frac{1}{kH}, \tag{3.12}$$

$$\frac{sin\big(k(z+H)\big)}{sin(kH)} = \frac{e^{k(z+H)} - e^{-k(z+H)}}{e^{kH} - e^{-kH}} = \frac{(1 + k(H+z) + \cdots) - (1 - k(H+z) + \cdots)}{(1 + kH + \cdots) - (1 - kH + \cdots)} \to \frac{2k(H+z)}{2kH} = 1 + \frac{z}{H}. \tag{3.13}$$

The shallow water waves are therefore described by:

$$\eta = A\,cos(kx - \omega t), \tag{3.14}$$

$$\omega = \sqrt{gH}k, \tag{3.15}$$

$$u = \frac{A\omega}{kH} cos(kx - \omega t), \tag{3.16}$$

$$w = A\omega\left(1 + \frac{z}{H}\right) sin(kx - \omega t). \tag{3.17}$$

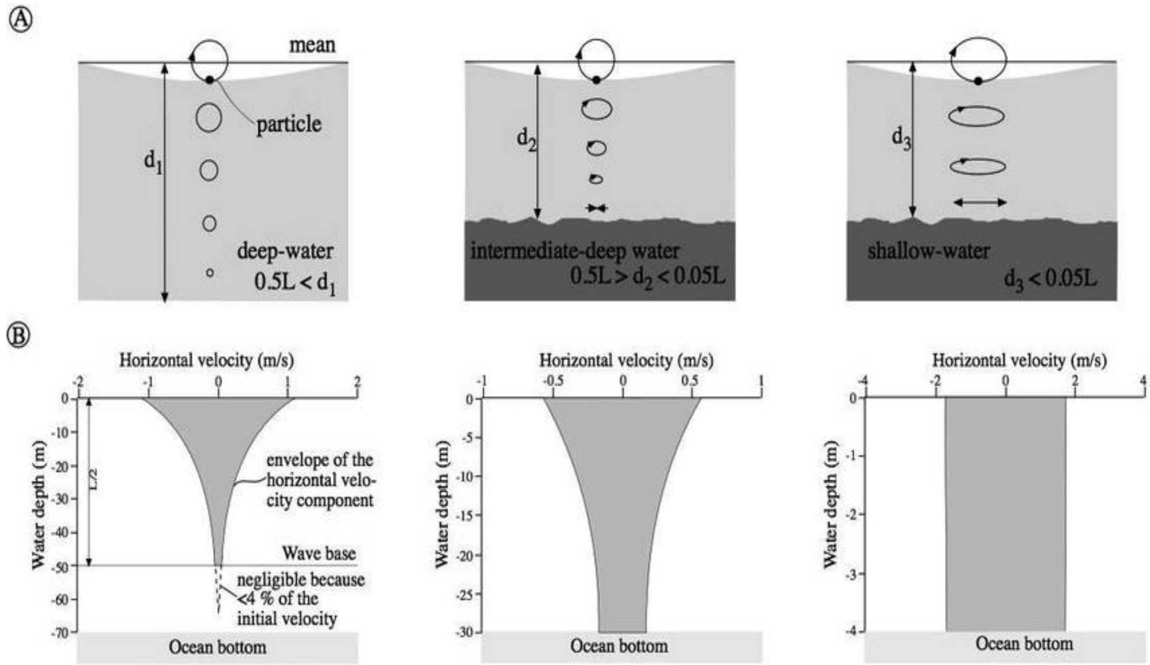

Figure 3-1:*Particle motion in deep water, intermediate deep water and shallow water and its velocity. Motion of the particle in the first image is circular, and radius of the motion decreases with depth. On the second image, at the ocean surface, motion is curcular, but as depth increases, the motion follows elliptic curve. For shallow water, shown at the last image, motion is elliptic, while at the bottom it becomes one dimensional (moving from the left to the right).*

On the Figure 3-1, we can see motion of a partcle in deep water, intermediate deep water and shallow water. Horizontal velocity for the case of deep water decreases almost exponentionaly with depth. After certain point, it can be neglected. For intermediate-deep water, velocity also decreases, but not as fast as in the case of deep water. For shallow water, horizontal velocity is constant with depth.

4 Derivation of the equations of motion

Fluid mechanic is a field theory. Typically, fields used to describe any fluid are pressure $p(x,y,z,t)$, mass density $\rho(x,y,z,t)$ and fluid velocity $\vec{v}(x,y,z,t)$. Distribution of mass in space is continuous.

Let n be the amount of X per unit volume. The amount of X is conserved, then the next equation is true for n:

$$\frac{\partial n}{\partial t} + \nabla\vec{F} = 0. \tag{4.1}$$

In the equation above, $\vec{F} = (F_x, F_y, F_z)$ is the flux of F. Using the definition of a flux, the equation can be rewritten as:

$$\frac{\partial n}{\partial t} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0. \tag{4.2}$$

If the velocity of X is $\vec{q}$, then equation can also be recast in following form:

$$\frac{\partial n}{\partial t} + \nabla(n\vec{q}) = 0. \tag{4.3}$$

In a fluid, velocity of mass is $\vec{v} = (u, v, w)$. Conservation law for mass is:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{v}) = 0. \tag{4.4}$$

Now, momentum per unit volume is $\rho\vec{v}$. Conservation can be applied to each component of momentum. Considering the x-component and the fact that force causes a change in the momentum, it can be written as:

$$\frac{\partial \rho u}{\partial t} + \nabla(\rho u\vec{v}) = f_x. \tag{4.5}$$

In this equation, f_x is the x-component of force per unit volume, f.

There are many different forces that can act on a fluid, but amongst those, pressure force of the fluid on itself is the one that is present in any situation. If a small cubic volume of a fluid is taken, then the pressure force per unit volume is $\vec{f} = -\nabla p$. The negative sign means that the fluid is being pushed away from a high pressure area to a low pressure area.

Taking all this into account, equation (4.5) becomes:

$$\frac{\partial(\rho u)}{\partial t} + \nabla(\rho u\vec{v}) = -\frac{\partial p}{\partial x}. \tag{4.6}$$

Changing x to y, and u to v, equation for y-component can be obtained:

$$\frac{\partial(\rho v)}{\partial t} + \nabla(\rho v\vec{v}) = -\frac{\partial p}{\partial y}. \tag{4.7}$$

For z-component, the equation has a small difference. There is an additional term, since gravity needs to be accounted for:

$$\frac{\partial(\rho w)}{\partial t} + \nabla(\rho w\vec{v}) = -\frac{\partial p}{\partial z} - \rho g. \tag{4.8}$$

If density of the fluid is constant, then:

$$\nabla\vec{v} = 0. \tag{4.9}$$

Equations (4.6), (4.7) and (4.8) simplify to:

$$\frac{\partial u}{\partial t} + \vec{v}\nabla u = -\frac{\partial p}{\partial x}, \tag{4.10}$$

$$\frac{\partial u}{\partial t} + \vec{v}\nabla v = -\frac{\partial p}{\partial y}, \tag{4.11}$$

$$\frac{\partial w}{\partial t} + \vec{v}\nabla w = -\frac{\partial p}{\partial z} - g. \tag{4.12}$$

These equations can now be written as a single vector equation:

$$\frac{D\vec{v}}{Dt} = -\nabla(p + gz). \tag{4.13}$$

Advective derivative operator $\frac{D}{Dt}$ was used to simplify the equations:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}\nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{4.14}$$

Usually, in any fluid, there is also viscosity. Viscosity does not contribute to the phenomena discussed here. All equations are ideal hydrodynamics equations.

Velocity is given as three independent fields, $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$. If the velocity is written using a single scalar field $\Phi(x,y,z,t)$:

$$\vec{v} = (u,v,w) = \nabla\Phi = \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right), \tag{4.15}$$

then equations:

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \qquad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \qquad \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \tag{4.16}$$

are true for potential flow. Potential flow describes the velocity field as the gradient of a scalar function: the velocity potential Φ. Now, if velocity can be rewritten using (4.15), equation (4.9) becomes Laplaces's equation:

$$\nabla^2 \Phi = 0. \tag{4.17}$$

Upon substituting Laplace's equation in (4.10), we get:

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial x} + \nabla\Phi \cdot \nabla\left(\frac{\partial \Phi}{\partial x}\right) = -\frac{\partial p}{\partial x}, \tag{4.18}$$

Rewriting $\nabla\left(\frac{\partial \Phi}{\partial x}\right)$ as $\frac{\partial}{\partial \mathrm{x}}\nabla\Phi$ yields:

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial x} + \nabla\Phi \cdot \frac{\partial}{\partial x}\nabla\Phi = -\frac{\partial p}{\partial x}, \tag{4.19}$$

further simplifying to:

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{1}{2}\nabla\Phi \cdot \nabla\Phi\right) = -\frac{\partial p}{\partial x}, \tag{4.20}$$

which results in:

$$\frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial \Phi}{\partial \mathrm{t}} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi + \mathrm{p}\right) = 0. \tag{4.21}$$

Similar to this, from equations (4.11) and (4.12) we get:

$$\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi + p\right) = 0, \tag{4.22}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi + p + gz\right) = 0. \tag{4.23}$$

As equations (4.10), (4.11) and (4.12) were written together as a single vector equation, these equations can also be written as a single equation:

$$\nabla\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi + p + gz\right) = 0. \tag{4.24}$$

Since the gradient of the prior function in parentheses is equal to zero, this means that the function is only a function of time:

$$C(t) = \frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi + p + gz. \tag{4.25}$$

If Φ is replaced by $\Phi + \int C(t)$, the hydrodinamic equations are:

$$\nabla^2 \Phi = 0. \tag{4.26}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla \Phi \cdot \nabla \Phi + p + gz = 0, \tag{4.27}$$

The first equation is, as mentioned, Laplace's equation. The second equation is the Bernoulli's equation. These are nonlinear differential equations and they are very difficult to solve. Usually, the method of solving consists of many approximations.

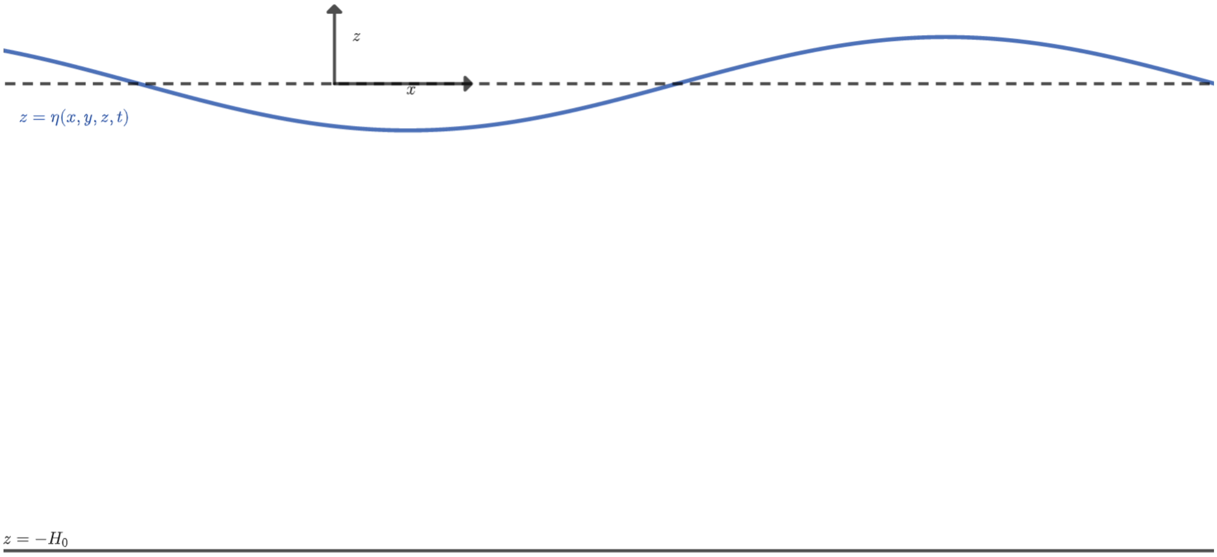

Figure 4-1: *Geometry for fundamental wave theory. Ocean bottom is considered flat with depth equal to $-H_0$. Blue curve represents surface distrurbance. Surface of the ocean is $\eta(x, y, t)$ and we are observing 2D system with x-axis and z-axis.*

Fundamental problem of the wave theory is now in order to be solved. Ocean with a flat bottom at $z = -H_0$ is considered. Surface of the ocean is $\eta(x, y, t)$. Movement of the fluid is described using Laplace's equation and Bernoulli's equation. Also, boundary conditions on the top and bottom of the ocean are needed. Bottom boundary condition is:

$$w = \frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = -H_0. \tag{4.28}$$

This boundary condition simply states that there is no flow through the ocean bottom.

There are two top boundary conditions. First condition, known as the kinematic boundary condition, says that the fluid particles on the free surface remain there:

$$\frac{D}{Dt}\big(z - \eta(x, y, t)\big) = w - \frac{D\eta}{Dt} = \frac{\partial \Phi}{\partial z} - \frac{\partial \eta}{\partial t} - \nabla \Phi \cdot \nabla \eta = 0 \ \text{ at } \\ z = \eta(x, y, t). \tag{4.29}$$

Second top boundary condition is a dynamic boundary condition. This condition states that the pressure is continuous on the free surface:

$$p = p_a \;\text{ at }\; z = \eta(x, y, t). \tag{4.30}$$

In this condition, p_a is atmospheric pressure. Two top boundary conditions say that the particles on a free surface and free surface have the same velocity and there is no pressure discontinuity that results in infinite acceleration of the free surface. Combining equations and boundary conditions, the following equations are obtained:

$$\nabla^2 = 0 \;\text{ on }\; -H_0 < z < \eta(x, y, t), \tag{4.31}$$

$$\frac{\partial \Phi}{\partial} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi + p + gz = 0 \;\text{ on }\; -H_o < z < \eta(x, y, t), \tag{4.32}$$

with the boundary conditions:

$$\frac{\partial \Phi}{\partial z}(x, y, -H_0, t) = 0, \tag{4.33}$$

$$\frac{\partial \Phi}{\partial z}(x, y, \eta(x, y, t), t) - \frac{\partial \eta}{\partial t} - \nabla\Phi(x, y, \eta(x, y, t), t) \cdot \nabla\eta(x, y, t) = 0, \tag{4.34}$$

$$p(x, y, \eta(x, y, t), t) = p_a(x, y, t). \tag{4.35}$$

From these equations, the boundary conditions seem rather complex. There are two unknowns: p and Φ, and location of those depends on η. In order to solve the problem, η must be known. Using linearization about the state of rest, this can be avoided.

Another problem is p_a at the surface. Atmosphere is the main generator of waves, but since its density is much lower than the density of the ocean, it can be replaced by vacuum. This means that the value of p_a is 0.

State of rest is $\Phi = \eta = 0$. If the waves that are present are very small, then they represent a small disturbance from the state of rest. In that case, Φ and η are very small and all terms proportional to products of Φ and η can be neglected (product of two small numbers is an even smaller number). To get a linear approximation of (4.34), Taylor expansion is used:

$$\frac{\partial \Phi}{\partial z}(x, y, \eta(x, y, t), t) = \frac{\partial \Phi}{\partial z}(x, y, 0, t) + \frac{\partial^2 \Phi}{\partial z^2}(x, y, 0, t)\eta(x, y, t) \approx \frac{\partial \Phi}{\partial z}(x, y, 0, t). \tag{4.36}$$

This means that the linear approximation of (4.32) is:

$$\frac{\partial \Phi}{\partial z}(x, y, 0, t) = \frac{\partial \eta}{\partial t}(x, y, t). \tag{4.37}$$

Linnear approximation of (4.33) is obtained when setting p_a to zero, since it yields $p = 0$ at $z = \eta(x, y, t)$. It is easier if this is also expressed over Φ and η, so by using (4.32), since it is true throughout the fluid, the following equation is obtained:

$$\frac{\partial \Phi}{\partial t}(x, y, 0, t) + g\eta(x, y, t) = 0. \tag{4.38}$$

When all this is done, solving the linear problem (LP) can begin. Linear equations are simpler and have certain appealing properties, such as superposition, that says that the sum of two solutions is also a solution.

First step is to eliminate η from the equation (4.38) to get the top boundary condition only in terms of Φ:

$$\frac{\partial^2 \Phi}{\partial t^2}(x, y, 0, t) + g\frac{\partial \Phi}{\partial z}(x, y, 0, t) = 0. \tag{4.39}$$

Because superposition principle holds, solutions in the following form are plausable:

$$\Phi(x, y, z, t) = F(z)\, sin(kx + ly - \omega t). \tag{4.40}$$

Arbitrary constants in this equation are k and l, while ω is constant that needs to be determined. $F(z)$ is a function that also needs to be determined. By substituting the above into $\nabla^2 = 0$, we get:

$$\frac{d^2 F}{dz^2} = \kappa^2 F, \tag{4.41}$$

where $\kappa^2 = k^2 + l^2$. This is a differential equation and its general solution reads:

$$F(z) = C_1' e^{\kappa Z} + C_2' e^{-\kappa Z}. \tag{4.42}$$

Equivalent form of this equation is:

$$F(z) = C_1\, cosh\big(\kappa(z + H_0)\big) + C_2\, sinh\big(\kappa(z + H_0)\big). \tag{4.43}$$

Now, the solution can be written in the form:

$$\Phi(x, y, z, t) = \{C_1\, cosh\big(\kappa(z + H_0)\big) \tag{4.44}$$

$$+ C_2\, sinh\big(\kappa(z + H_0)\big)\}\, sin(kx + ly - \omega t).$$

Inserting this into the boundary condition $\frac{\partial\Phi}{\partial z}(x, y, -H_0, t) = 0$, yields:

$$(C_1\kappa\, sinh(0) + C_2\kappa\, cosh(0))\, sin(kx + ly - \omega t)$$
$$= C_2\kappa\, sin(kx + ly - \omega t) = 0. \tag{4.45}$$

.

This equation must be true for all possible (x, y, t) and in order to be so, constant C_2 must be zero. Now, the solution is:

$$\Phi(x, y, z, t) = C_1\, cosh\big(\kappa(z + H_0)\big)\, sin(kx + ly - \omega t). \tag{4.46}$$

By substituting this form of the solution into the boundary condition $\frac{\partial^2\Phi}{\partial t^2}(x, y, 0, t) + g\frac{\partial\Phi}{\partial z}(x, y, 0, t) = 0$, we get:

$$-\omega^2 C_1\, cosh(\kappa H_0) + g\kappa C_1\, sinh(\kappa H_0) \;=\; 0. \tag{4.47}$$

There is one obvious solution to this equation, and that is putting $C_1 = 0$ and getting $\Phi = 0$. But that is not the wanted solution. Assuming $C_1 \neq 0$ yields:

$$\omega^2 = g\kappa\, tanh(\kappa H_0). \tag{4.48}$$

Remembering Postulate 1, it can be seen that this is the general dispersion relation for ocean waves.

Combining (4.37) and (4.46):

$$\eta = -\frac{1}{g}\frac{\partial\Phi}{\partial t}(x, y, 0, t) = \frac{\omega C_1}{g} cosh(\kappa H_0)\, cos(kx + ly - \omega t). \tag{4.49}$$

Combining (4.15) and (4.46):

$$u = \frac{\partial\Phi}{\partial x}(x, y, z, t) = kC_1\, cosh\big(\kappa(z + H_0)\big)\, cos(kx + ly - \omega t), \tag{4.50}$$

and

$$w = \frac{\partial\Phi}{\partial x}(x, y, z, t) = \kappa C_1\, sinh\big(\kappa(z + H_0)\big)\, sin(kx + ly - \omega t). \tag{4.51}$$

By setting amplitude A to be equal to:

$$A = \frac{\omega C_1}{g} cosh(\kappa H_0) = \frac{\kappa C_1}{\omega} sinh(kH_0), \tag{4.52}$$

last three equations become:

$$\eta = A\,cos(kx + ly - \omega t), \tag{4.53}$$

$$u = A\omega \frac{k}{\kappa} \frac{cosh\big(\kappa(z + H_0)\big)}{sinh(\kappa H_0)} cos(kx + ly - \omega t), \tag{4.54}$$

$$w = A\omega \frac{sinh\big(\kappa(z + H_0)\big)}{sinh(\kappa H_0)} sin(kx + ly - \omega t). \tag{4.55}$$

5 Conclusion

In this thesis, ocean surface waves were described. Two postulates that describe the general form of the wave and allow us to add them, were used as a foundation for the derivation of the equations of motion. Additionaly, Laplace's equation and Bernoulli's equation, two important equations in fluid mechanics, were aslo used. Those are nonlinear equations, and in order to solve them, we used approximations. These approximations simplify the problem and allow us to explore the linear wave theory.

Atmosphere is the main generator of ocean surface waves, but since its density is much lower than the density of the ocean, it can be replaced by vacuum. This gives us one of the approximations. By using Taylors expansion, we eliminated η, the wave equation, from one of the boundary conditions. This was done in order to get the linear equations, to simplify the process. One property of linear equations that was used is superposition. By using this principle, we wrote the general form of the solution. Although this method did ignore some of the fluid properties, such as viscosity, it is successful in delivering plausable equations of motion. To get the full description, non-linear theory should be fully explored.

6 Literature

[1]R. Salmon , *Introduction to Ocean Waves*, University of California, San Diego

[2]B. Cushman-Roisin and J. Beckers (2008). *Introduction to Geophysical Fluid Dynamics Physical and Numerical Aspects,* Academic Press

[3]S. Pond and G. L. Pickard, *Introductory Dynamical Oceanography* (2nd ed.), University of British Colombia, Vancouver

[4]J.A. Knauss (1997). *Introduction to physical oceanography* (2nd ed.), University of Rhode Island